Direct Update of Dynamic Mathematical Models from Modal Test Data

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Two methods for the direct updating of mathematical models based on modal test data are described. A set of minimum required constraints derived from eigendynamic and force equilibrium conditions are presented. Both methods generate mass and stiffness matrices fitted exactly to the given modal test data. One method changes all coefficients of the mass respective to the stiffness matrix and lessens the solution effort by the algebraical elimination of the Lagrangian multipliers. The other method changes only those selected matrix coefficients requiring additional solutions for the numerical calculation of the Lagrangian multipliers. Examples and comments on the applicability of the methods are given.

Introduction

SPACECRAFT structure development relies heavily on analysis results obtained from mathematical dynamic models. This means that model verification is essential once the hardware is available and can be tested. In the case of insufficient correlation between analysis and test data, update strategies must be applied to obtain modified mathematical models within the required accuracies.

Two categories of update/indentification methods are presented in the literature:

1) Update methods using dynamic mathematical models of structures to improve them on the basis of measured data. These methods use different mathematical formulations such as a) linear least squares formulations for direct matrix updating[1-8] and b) least squares/sensitivity strategies.

2) Direct identification methods using test data only. These methods use time domain or frequency domain test data to construct modal and physical mathematical models of a structure.

A comprehensive survey and summary of these methods is given in Ref. 9.

In the following, two methods for direct mass and stiffness matrix updating of category 1a are discussed. The first method, called direct matrix update (DMU), is based on the methods originally published by Baruch and Bar-Itzhack[1] and Berman and Nagy.[2] The modifications described in this paper permit the update of free/free, as well as fixed/free, structure models satisfying force equilibrium or total mass in addition to orthogonality and eigendynamic constraints. A second variant is described that permits the arbitrary selection of coefficients as variables of the matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ to overcome some drawbacks of the DMU method.[4-7]

Direct Matrix Update (DMU)

The direct matrix update (DMU) method uses mass and stiffness matrices $\boldsymbol{M}_A$ and $\boldsymbol{K}_A$ from the analysis assumed to be incorrect and test data assumed to be correct. (Subscript A means analysis and T test.) The basic equations are given in Tables 1–3 and discussed in the following. All coefficients of the matrix $\boldsymbol{M}_A$ are modified with minimum variation to satisfy a set of linear equality constraints. Basic constraints are given in Table 2. In a second step, the stiffness matrix $\boldsymbol{K}_A$ is updated using the already updated mass matrix $\boldsymbol{M}$. The objective function of minimum variation ensures that the changes of all matrix coefficients are minimized in a least squares sense, so that the updated matrices should still be representative of the physical structure described by the analytical model. The constraints are considered via Lagrangian multipliers that are eliminated algebraically, leading to a closed-form solution. This algebraic elimination reduces the numerical solution effort. On the other hand, it makes the method less flexible for the introduction of new types of equality constraints.

In order to control not only the eigendynamics but also other quantities such as total mass, effective masses, or interface forces, additional constraints are introduced as given in Table 2. The objective functions and problem-oriented variations of the constraint equations for the mass and stiffness matrix updates are given in Tables 1 and 3 together with the final update equations. There is still a closed-form solution for the updated matrices that requires no additional or even iterative eigendynamic analyses. The principles of their derivation are given in Ref. 1. The matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ fulfill the equality constraints exactly.

Mass Matrix Update

According to Table 1, this update can be performed in different ways, while the final equation for the updated $\boldsymbol{M}$ stays the same. The different constraints used for the update are introduced via different meanings of $\boldsymbol{X}_c$ and $\boldsymbol{M}_c$, depending on the test data available:

1) Using the eigenmodes $\boldsymbol{X}_c = \boldsymbol{X}$ of a free/free system. $\boldsymbol{X}$ includes the rigid-body modes. Required accurate test results are eigenmodes $\boldsymbol{X}$, total mass (rigid-body mass) $\boldsymbol{M}_R$, and generalized mass $\boldsymbol{M}_G$. All mass informations are stored in $\boldsymbol{M}_c$.

2) Using in a first step eigenmodes $\boldsymbol{X}_c = \boldsymbol{X}$ of a fixed/free system together with the generalized mass matrix $\boldsymbol{M}_G$ and in a second step the rigid-body modes ($\boldsymbol{X}_c = \boldsymbol{X}_R$) to control the total mass $\boldsymbol{M}_R$. The rigid-body modes are calculated from the measuring point coordinates. The two steps are repeated iteratively until the final mass matrix change is less than a defined limit.

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3) Using a combination of fixed/free eigenmodes and rigid-body modes with $X_c = [X_R, X]$. The rigid-body modes X_R are calculated from the measuring point coordinates and the fixed/free eigenmodes X are expanded by zero values at the supported degrees of freedom. Required are correct eigenmodes X, generalized masses M_G, the total mass M_R, and the participation factors M_p. The participation factors are a measure for the modal reaction forces at the fixed interface of the test structure.

Stiffness Matrix Update

This update is performed according to Table 3 with the updated mass matrix M and constraints 1-3 and 5 taken from Table 2. Again the combination of X_R and X in X_c is used to calculate the final stiffness matrix in one step. The eigenmodes X are expanded by zero terms to include the degrees of freedom (DOF) of the supports. The modal interface forces at the supports should be measured. They may be calculated using a sufficient accurate mass matrix. For isostatic supports, these modal interface forces are directly calculated from the participation factors M_p times the eigenvalues ω^2.

If the analytical stiffness matrix K_A seems to be more accurate than the mass matrix M_A, one can start the update process with K_A, applying constraints 1 and 2 of Table 2. In a second step, M_A would be updated applying constraints 1, 2, and 4, of Table 2. Such variants and applications of the update equations are given in Ref. 8.

One should be aware that the simplicity of the method is offset by such drawbacks as the destruction of the bandwidth of previously banded matrices and very limited control of the percentage change in the individual matrix coefficients. These drawbacks are circumvented by the generalization discussed in the following sections.

Generalized Linear Least Squares Formulation

The optimization problem of updating the structure system matrices M_A and K_A can be solved in a more general way.[4-7] To perform the simple mathematical formulations, the following definitions are made:

1) K_i are the arbitrarily selected variable coefficients of the stiffness matrix to be updated, written in vector form.

2) n is the total number of coefficients K_i.

3) $\bar{K}_i$ are the starting values for variables K_i, in general taken from a finite-element stiffness matrix K_A.

Now, the objective function can be written as

$$\text{Minimize } \epsilon = \sum_{i=1}^{n} W_i (K_i - \bar{K}_i)^2 \tag{1}$$

where W_i is the weighting factor, e.g., $W_i = 1/\bar{K}_i^2$ for percentage changes of K_i.

The equality constraints are given by

$$\left(\sum_{i=1}^{n} A_{ji} K_i\right) - R_j = 0 \text{ with } j = 1 \ldots r \tag{2a}$$

where r is the number of constraint equations.

In matrix formulation, Eq. (2a) becomes

$$AK - R = 0 \tag{2b}$$

All of the constraints noted in Table 2 can be rewritten in this form. Any constraint in a linear equality form can be introduced. With this formulation of objective function and constraints, the arbitrary selection of the coefficients as variables and the individual weighting of the variables becomes possible. In the update process a banded matrix structure can thus be kept, substructures can be selected using corresponding local modes, or artificially banded

Table 2 Equality constraints

1) Symmetry		$K - K^T = 0$
	or	$M - M^T = 0$
2) Orthogonality		$X^T K X - \omega^2 = 0$
	or	$X^T M X - M_G = 0$
		with $M_G = I$
3) Eigendynamics for K update, M known		$KX - MX\omega^2 = 0$
4) As constraint 3 for M update, K known		
5) Force equilibrium		$KX_R = 0$
6) Rigid-body mass		$X_R^T M X_R - M_R = 0$
7) Participation factor		$X_R^T M X - M_p = 0$

Table 1 Direct mass matrix updating

Objective function

$$\epsilon = \min \| M_A^{-1/2} (M - M_A) M_A^{-1/2} \|$$

Constraints

$$M^T - M = 0$$

$$X_c^T M X_c - M_c = 0$$

where

X_c = rigid-body modes X_R or eigenmodes X or combination $[X_R | X]$

M_c = rigid-body mass M_R or generalized mass M_G or combination of both with participation factors M_p, with

$$M_c = \begin{bmatrix} M_R & M_p \\ M_p & M_G \end{bmatrix}$$

Final equation

$$M = M_A + M_A X_c M_{Ac}^{-1} (M_c - M_{Ac}) M_{Ac}^{-1} X_c^T M_A$$

where

$$M_{Ac} = X_c^T M_A X_c$$

Similar equations for K (see Table 3 and Ref. 9)

Table 3 Direct stiffness matrix update

Objective function

$$\epsilon = \min \| M^{-1/2} (K - K_A) M^{-1/2} \|$$

Constraints

1) $K^T - K = 0$

2) $X_c^T K X_c - \omega_c^2 = 0$

where

$$X_c = [X_R, X]$$

$$\omega_c^2 = \text{diag}(0, \ldots, 0, \omega_1^2, \ldots, \omega_n^2)$$

3) $KX_c - MX_c\omega_c^2 + F_s = 0$

including eigendynamic and force equilibrium constraints, with F_S = modal reaction forces at supported DOF's.

Final equation

$$K = K_A + \Delta + \Delta^T$$

where

$$\Delta = \{ -K_A X_c - F_s + MX_c [\omega_c^2 - 0.5 M_c^{-1} (\omega_c^2 - X_c^T K_A X_c)] \} M_c^{-1} X_c M$$

matrices can be constructed after the band has been destroyed by a condensation process in the finite-element analysis. Matrix symmetry is directly handled using only the upper triangle of the symmetric matrices $\boldsymbol{M}$ and $\boldsymbol{K}$, leading to considerable savings in computational time and storage requirements.

The problem is solved via Lagrangian multiplier λ with

$$\psi = \sum_{i=1}^{n} W_i (K_i - \bar{K}_i)^2 + \sum_{k=1}^{r} \lambda_k \left(\sum_{i=1}^{n} A_{ki} K_i - R_k \right) \tag{3}$$

which has to satisfy at the solution

$$\frac{\partial \psi}{\partial K_i} = 2W_i (K_i - \bar{K}_i) + \sum_{k=1}^{r} \lambda_k A_{ki} = 0 \tag{4}$$

for $i = 1 \ldots n$, resulting in

$$K_i = \bar{K}_i - \left(\sum_{k=1}^{r} \lambda_k A_{ki} \right) \cdot \frac{1}{2} W_i \tag{5a}$$

or in matrix formulation

$$K = \bar{K} - \tfrac{1}{2}(W^{-1} A^T \lambda) \tag{5b}$$

where W is the diagonal matrix. Substitution of Eq. (5) in Eq. (2) gives

$$A[K - \tfrac{1}{2}(W^{-1} A^T \lambda)] - R = 0 \tag{6}$$

with

$$\bar{R} = A\bar{K} - R \tag{7}$$

and

$$L = \tfrac{1}{2}(A W^{-1} A^T) \tag{8}$$

The final equation for the determination of the Lagrangian multipliers becomes

$$L\lambda = \bar{R} \tag{9}$$

The constraint matrix A as well as the Lagrange matrix L may have a rank smaller than the matrix size. So, Eq. (9) can be solved by Gaussian elimination and rank determination of L, as well as by calculation of the pseudoinverse of L using singular value decomposition and least squares solutions,[10] giving

$$\lambda = L^I \bar{R} \tag{10}$$

with the final update equation

$$K = \bar{K} - W^{-1} A^T (A W^{-1} A^T)^I \bar{R} \tag{11}$$

The maximum number of independent constraint equations becomes

$$m(m+1)/2 \tag{12}$$

for the orthogonality constraints with m the number of modes and

$$km - m(m-1)/2 \tag{13}$$

for eigendynamic constraints with k the number of DOF's. The orthogonality constraints are linearily dependent on the eigendynamic constraints.

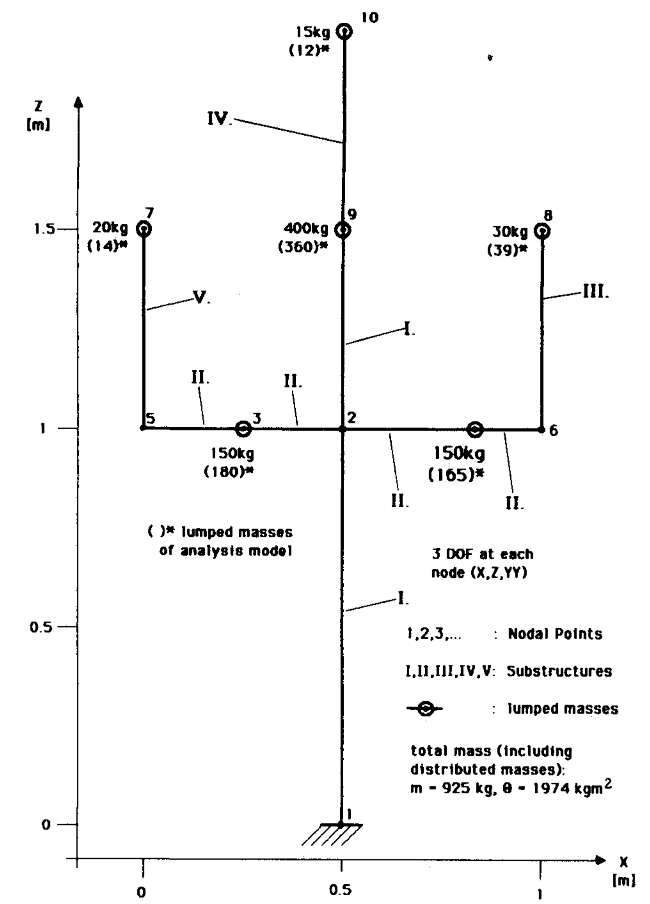

Fig. 1 Simulated test model (27 DOF).

Table 4 Eigenfrequencies and effective masses of "test" and analysis (27 DOF model)[a]

No.	f_T^*, Hz	f_A^*, Hz	m_{X_T}, kg	m_{X_A}, kg	m_{Z_T}, kg	m_{Z_A}, kg
1	22.7	21.0	128.3	191.6	21.0	13.1
2	25.1	24.2	24.6	16.7	88.5	113.6
3	28.4	29.0	24.5	53.4	0.1	0.4
4	37.7	32.8	15.8	6.6	609.5	329.0
5	39.9	36.5	544.7	471.1	15.8	1.2
6	43.8	39.4	15.9	0.4	9.3	418.8
7	44.7	44.5	21.8	58.8	0.0	0.3
8	49.1	45.9	0.9	2.2	165.7	12.2
9	51.5	51.3	11.2	0.0	1.1	24.2
10	59.0	58.9	0.7	0.0	4.4	3.4

[a] T = "test," A = analysis before updating.

The same equations are valid for the mass matrix update. Thus, both mass and stiffness matrix improvements are possible.

Apart from using only a small number of update variables, the generalized linear least squares (GLLS) method requires more computational time than DMU because of the numerical treatment of the Lagrange variables.

Application of DMU to 27 DOF Model

DMU is applied to a small two-dimensional finite-element with 27 DOF (see Fig. 1). Its main characteristics, typical for spacecraft structures are large variations in local masses as well as a high density of frequencies in the range of 20–60 Hz and significant dynamic coupling in the forced response for base excitation.

Table 5 Comparison of eigenfrequencies and correlation of modes (27 DOF model)

No.	"Test"	Analysis	Eigenfrequencies, Hz Run 1	Run 2	Run 3	Run 4
1	22.7	21.0(1+)[a]	22.7(1+)	22.7(1+)	22.7(1+)	22.7(1+)
2	25.1	24.2(2+)	25.1(2+)	25.1(2+)	25.1(2+)	24.1(3+)
3	28.4	29.0(3+)	28.4(3+)	29.3(3+)	29.0(3+)	25.1(2+)
4	37.7	32.8(4−)	37.7(4+)	37.7(4+)	37.7(4+)	37.7(4+)
5	39.9	36.5(5)	39.9(5+)	39.9(5+)	39.9(5+)	39.9(5+)
6	43.8	39.4(6)	43.8(6+)	43.7(9)	42.3(9+)	42.0(9)
7	44.7	44.5(9)	44.7(7+)	43.8(6+)	43.8(6+)	43.8(6+)
8	49.1	45.9(7)	49.1(8+)	45.2(7−)	44.5(7−)	44.5(7−)
9	51.5	51.3(−)	51.5(9+)	50.1(−)	49.1(−)	48.4(7−)
10	59.0	58.9(10+)	59.0(10+)	58.9(10+)	58.8(10+)	58.7(10+)

[a] (n) = correlates with test mode n, $(n+)$ = very good correlation, $(n-)$ = bad correlation, $(-)$ = no correlation.

Table 6 Total mass influenced by updating (27 DOF model)

Model/run	M_X, kg	M_Z, kg	Θ_Y, kg·m^2
"Test"	925.0	925.0	1974.0
Analysis	940.7	940.7	1928.2
Run 1	976.2	940.8	2011.1
Run 2	976.2	950.1	2008.4
Run 3	916.8	928.7	1959.0
Run 4	925.0	925.0	1974.0

Different input data for the finite-element analysis are taken to construct the simulated "test" and the analysis models. The differences in stiffnesses and masses vary between ±40% and are distributed over the whole structure.

Table 4 shows the eigenfrequencies and effective masses (M_p^2/M_G) of the "test" and analysis models before updating (M_p is the participation factor, M_G the generalized mass according to Table 2).

The following update runs are performed with the DMU method:

1) Then eigenfrequencies, mass matrix update with orthogonality, and symmetry constraints.

2) Five eigenfrequencies (1, 2, 4–6), mass matrix update constraints as for run 1.

3) Five eigenfrequencies as in run 2, mass matrix update iteratively with total mass constraint in the first step and orthogonality in the second step, five iterations.

4) Five eigenfrequencies as in run 2, mass matrix update in one step with orthogonality, total mass and participation factor constraints simultaneously.

For runs 2–4, it is assumed that the third eigenfrequency has not been measured. The stiffness matrix is updated in these four runs using symmetry and eigendynamic constraints.

Table 5 shows that the updated eigenfrequencies are equal to the required ones, meaning they are equal to those eigenfrequencies of the "test" used for the update run. Other eigenfrequencies not taken into account, such as eigenfrequency $f_3 = 28.4$ Hz in runs 2–4, may be shifted to wrong values.

Table 6 shows the change of the total mass. The results make obvious that the introduction of a total mass constraint is mandatory to avoid significant and physical meaningless mass changes.

Table 7 shows the effective masses for base excitation in the z direction. Even without the participation factor and/or total mass constraints, the effective masses converge to reasonable values (e.g., run 1, 10 modes).

Figure 2 shows the influence of DMU updating on the frequency response. Since the updated matrices satisfy exactly the requirements given in the equality constraints, the improvement in the response based on the modal data is significant. Small discrepancies between the updated and "test" responses are referred to modes 3 and 7, as well as higher modes not taken into account in update run 4.

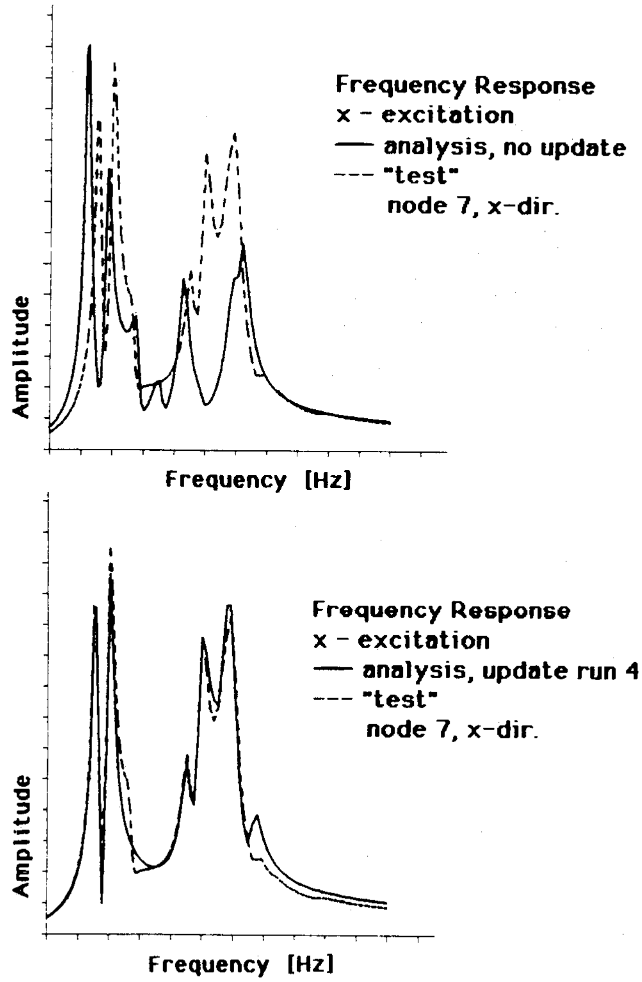

Fig. 2 Comparison of frequency response (27 DOF model).

From the update results one can draw the following conclusions:

1) It is mandatory to apply the total mass constraints to avoid meaningless mass changes.

2) All modes X_T and eigenfrequencies f_T in the frequency range of interest must be available to avoid uncontrolled frequency shifts (spurious modes).

3) It is possible to improve the effective masses without the participation factor constraints. This may support the update analysis if measured interface forces are not available.

Table 7 Comparison of effective masses in Z direction (27 DOF model)

	Effective masses in Z direction, kg					
Mode	"Test"	Analysis	Run 1	Run 2	Run 3	Run 4
1	21.0	13.1	20.6	24.5	22.3	21.0
2	88.5	113.5	87.8	58.3	57.9	0.0[a]
3	0.1	0.4	0.1	0.1	0.1	88.5
4	609.5	329.0	619.4	651.2	636.9	609.5
5	15.8	1.2	16.1	13.6	11.8	15.8
6	9.3	418.8	9.8	4.6	6.1	2.6
7	0.0	0.3	0.0	25.4	24.7	9.3
8	165.7	12.2	172.1	63.6	79.9	68.2
9	1.1	24.2	1.0	95.4	77.4	98.8
10	4.4	3.4	4.3	6.2	4.6	4.3
Σ	915.5	916.2	931.1	942.8	921.7	918.0

[a]Significant frequency shift of mode 3. Mode 3 was not used for update runs 2-4.

4) The DMU update can be easily applied without detailed knowledge about the correlation between test and analysis modes or the location of error sources in the analysis model.

Application of GLLS

The GLLS is applied to an example with 16 nonzero stiffness coefficients as given in Refs. 5 and 6 and shown in Fig. 3. Valuable numerical results describing the influence of measuring errors of mode shapes on the update results as well as the application of static test data are given in Ref. 6. The solution method of Refs. 5 and 6 is included in GLLS, if only the nonzero terms of $\bar{K}$ are changed. Weighting is performed with $W_i = 1/\bar{K}_i^2$ to minimize the percentage change of the variables.

The number of variable stiffness coefficients is $n = 16$. The number of linear independent constraint equations r depends on the number of "test" eigenmodes m applied for the update and is calculated according to Eq. (13). The sensitivity of the update results to changes of the "test" eigenfrequencies is extreme in the case of the overdetermined equation system (see Table 8, $r > n$, $m = 3$). The update results become less sensitive to input inaccuracies for

Table 8 GLLS application to 8 DOF model using three modes

			Adjusted stiffness coefficients		
Coeff. loc.	Corrupted stiffness coefficient	Exact stiffness values	Frequency input, 6 digits	Frequency input, 4 digits	Frequency input, 1% error
1,1	2.00	1.50	1.50	1.58	−23.85
1,2	−2.00	−1.50	−1.50	−1.75	75.41
2,2	1512.00	1011.50	1011.50	1012.26	764.18
2,3	−10.00	−10.00	−10.00	−9.95	−18.25
3,3	1710.00	1110.00	1110.00	1109.54	1167.39
3,5	−200.00	−100.00	−100.00	−99.62	−144.96
4,4	850.00	1100.00	1100.00	1099.55	1161.49
4,5	−200.00	−100.00	−100.00	−99.63	−142.15
4,6	−200.00	−100.00	−100.00	−99.56	−164.17
⋮	⋮	⋮	⋮	⋮	⋮
3 modes	$f_1 = 4.88052$ Hz $f_2 = 5.04691$ Hz $f_3 = 5.05526$ Hz		16 stiffness coefficients 21 constraint equations		

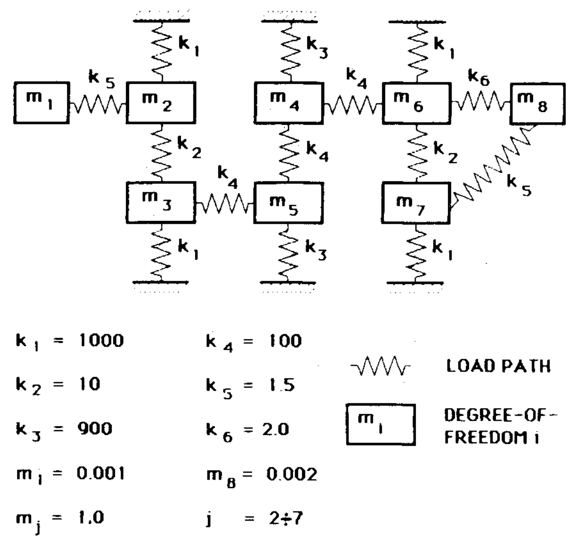

Fig. 3 Eight DOF spring-mass model.

Table 9 GLLS application to 8 DOF model using one and two modes

			Adjusted stiffness coefficients			
			Two modes		One mode	
Coeff. loc.	Corrupted stiffness coefficient	Exact stiffness values	Frequency input, 6 digits	Frequency input, 1% error	Frequency input, 6 digits	Frequency input, 1% error
1,1	2.00	1.50	1.50	1.47	1.73	−1.71
1,2	−2.00	−1.50	−1.50	−1.46	−2.10	−2.11
2,2	1512.00	1011.50	1011.50	991.17	1014.12	995.57
2,3	−10.00	−10.00	−10.00	−9.76	−10.15	−10.15
3,3	1710.00	1110.00	1110.00	1087.29	1274.66	1257.62
3,5	−200.00	−100.00	−100.00	−97.58	−197.91	−198.84
4,4	850.00	1100.00	1100.00	1077.53	1217.66	1201.37
4,5	−200.00	−100.00	−100.00	−97.58	−159.29	−161.09
4,6	−200.00	−100.00	−100.00	−97.58	−197.73	−198.66
⋮	⋮	⋮	⋮	⋮	⋮	⋮
8,8	6.0	3.5	4.35	4.31	5.19	5.16
			16 stiffness coefficients 15 constraint equations		16 stiffness coefficients 8 constraint equations	

undetermined equation systems (see Table 9, $r<n$, $m=2$). The convergence toward the correct stiffness values is promising. The results correlate well with those given in Ref. 6.

Correlation of Analysis and Test Data

The update procedures described above require reliable test data. To increase the accuracy and reliability of test data, it is not only necessary to improve the measurement methods. The test performance must be supported systematically by the analysis. Providing analysis data to support modal tests, some basic points to be considered are:

1) The degrees of freedom of the analysis should be compatible in terms of geometry, coordinate systems, and number of degrees of freedom with the test measurement channels.

2) The forces at the support points of the test structure should be measured.

3) Correlation of the analyses and tests should be performed on-line with the test.

Only compatible analysis and test models permit correlation within a short time. Force measurements permit the comparison of participation factors, which are important modal parameters to guarantee the correct interface forces in the coupled dynamic analysis and which are necessary to calculate effective masses from pure test data. Correlation on-line with the test using modal scale factors, modal assurance criterion, different variants of mode plots, effective masses, and orthogonality checks (to outline only the most important checks) helps to find any weaknesses during modal test performance. Bad correlation is the request to search for its sources. The updating process cannot be started until an intensive check of the test data results in a set of selected reliable data.

Several applications of the two update methods described above showed that a complete input data set containing all modes in the interesting frequency range is mandatory to achieve physical meaningful results. Thus, the test data set may be mixed with analytical data in the case of incomplete test results.

Conclusion

Although the update methods described in this paper have been sufficiently developed, additional investigations are required in the field of correlation and real hardware applications. Detailed correlation procedures are needed as preprocessors to establish a set of reliable test data or combined test/analysis data and as postprocessors to verify the increase in the accuracy and physical validity of the updated model.

Actually, considerable effort has been put into the development and improvement of procedures to update such design parameters as cross-section properties and mass densities. Part of that work was and still is the development of sensitivity analysis software for finite-element packages. Such work must be accompanied by error localization procedures, so that the selected design parameters to be updated are relevant to the problem solution. Also, the introduction of inequality constraints for the definition of the upper and lower bounds of the design parameters seems to be essential to receive physical meaningful values.

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